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LARGE DEFLECTIONS OF BUCKLED BARS UNDER DISTRIBUTED AXIAL LOAD†

D. E. PANAYOTOUNAKOS and P. S. THEOCARIS
Department of Theoretical and Applied Mechanics, The National Technical University of Athens, Athens, P.O. Box 77230, Greece (175-10)

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Abstract—An analysis of the strongly non-linear differential equations governing the elastica configuration of a flexible straight bar due to a longitudinal compressive force continuously distributed along its length, is presented in this paper. For the derivation of a straightforward closed-form solution of this complicated problem several functional transformations are used and a quantitative analysis is developed yielding reliable results in conformity with the physical problem. The functional transformations introduced in this paper though simple are original and useful in solving, without the use of numerical procedures, problems of non-linear elasticity.

1. INTRODUCTION

In the linear buckling theory (second-order theory), if longitudinal compressive forces are continuously distributed along a straight bar, the differential equation of the deflection curve of the buckled bar is no longer expressed by an equation with constant coefficients given by Timoshenko and Gere (1961). The solution of this equation requires the application of infinite series, or the recourse to one of the approximate methods, such as the energy method.

The problem of linear buckling of a straight prismatic bar due to its own weight was discussed first by Euler, who however, did not succeed in obtaining a satisfactory solution. This problem was solved by Greenhill (1884). A variety of linear buckling problems was indicated in this paper, which can be solved by using Bessel functions. Independently, the same problem was discussed in a complete manner by Jasinsky (1902), Dondorff (1907) and Kàrmàn and Biot (1940).

On the other hand, the problem of non-linear buckling analysis (elastica problem) of a straight bar was first solved in a complete form by Love (1944). He determined, as a first application of the theorem of Kirchhoff's kinetic analogues, the forms in which a straight thin rod can be held by forces and couples applied at its ends only. Also, analytic solutions of the strongly non-linear differential equations, governing the buckling, or post-buckling behaviour for straight and prismatic bars, due only to concentrated loads and couples, have been given by Frisch-Fray (1962) and Griner (1984). Especially, Griner (1984) has developed a parametric solution of the elastica problem concerning a thin, simply supported rod, subjected to terminal compressive forces and couples.

Large flexural deflections of bars were also solved (in terms of elliptic functions) when the loads consist of a combination of end forces and couples, as well as uniformly distributed transverse loads. These solutions were included in the treatise of Halphen (1888) and used in the paper by Panayotounakos and Theocaris (1986). Finally, the problem of either linear or non-linear buckling analysis in planar curved bars under uniform pressure or terminal concentrated forces is discussed in Panayotounakos and Theocaris (1981) and DaDeppo and Schmidt (1974).

The object of this paper is to present a closed-form solution for the problem of non-linear and buckling analysis for a straight prismatic bar due to a longitudinal compressive force, continuously distributed along its length. This problem was posed long ago, but its importance in applications to special engineering structures such as pipes, drills, etc. has only recently been realized. In this context it is worthwhile underlining that analytic solutions

† All correspondence related to this paper should be addressed to Prof. P. S. Theocaris, Department of Engineering Sciences, Athens National Technical University, P.O. Box 77230, Athens Gr175-10, Greece.

based on the non-linear theory of elasticity are impossible, because closed-form solutions of the governing non-linear differential equations have not been as yet developed.

The analysis was based on the integration of the exact, strongly non-linear, equilibrium differential equation for the values of the slope θ of the deflected elastica lying in the limited interval [0°, 20°]. The set of values of the slope was restricted in the interval [0°, 20°] because this interval is of practical interest for engineering structures and furthermore the approximate substitution of the appearing function cot θ, by $1/\theta$, is accurate enough for this interval and for any practical application. Then the solutions are derived through the reduction of the aforementioned differential equation to a final integrable form based also on further successive functional transformations.

It has been shown that the relations giving the critical load are divided into two terms; the first is in accordance with the corresponding buckling load in linear theory, while the second term depends on the free end slope of the deflected elastica.

Extending the proposed methodology for the interval [0°, 20°] and using new functional transformations, we succeeded also in reducing the same strongly non-linear equilibrium differential equation in simple integrable forms for the values of the slope θ lying in the subintervals (20°, 45°) and [45°, 90°]. For this purpose substitutions of the function cot θ were now introduced which were also accurate enough in these intervals. Consequently, it becomes obvious that the elastica configuration of the bar in all the subintervals of the slope cannot be described by a unique solution.

2. FORMULATION OF THE NON-LINEAR DIFFERENTIAL EQUATIONS

Consider the problem of large deflections of a straight prismatic bar fixed at its base and free at its upper end. The bar buckles under the action of a longitudinal compressive force q, which is uniformly distributed along the length l (Fig. 1). Such a type of loading is the bar's own weight. If the bar buckles, as shown by the dotted line in Fig. 1, the exact differential equation of the deflection curve is expressed by

$$\theta' = A\int_{x}^{x_a} (\eta - y)\, d\xi \tag{1}$$

in which

$$A = q/EJ \tag{2}$$

where θ is the slope of the deflected elastica; EJ the flexural rigidity of the structure and

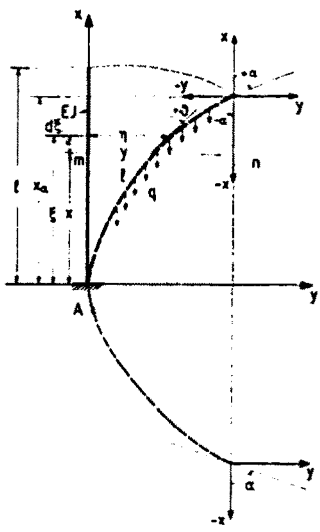

Fig. 1. Geometry and sign convention of a flexible straight bar under a uniformly distributed axial load.

primes denote differentiation with respect to the arc s. The integral on the right-hand side of eqn (1) represents the bending moment at any cross-section, mn, produced by the uniformly distributed load of intensity q (Timoshenko and Gere, 1961). Finally, the change in length of the column due to compression is neglected. This assumption is justified for the usual structural materials.

Differentiating eqn (1) once with respect to s and taking into account the well-known relations

$$\mathrm{d}x = \mathrm{d}s \cos\theta; \quad \mathrm{d}y = \mathrm{d}s \sin\theta \tag{3}$$

we obtain

$$\theta'' = A\frac{\mathrm{d}}{\mathrm{d}y}\left[\int_x^{x_a} (\eta - y)\,\mathrm{d}\xi\right]\frac{\mathrm{d}y}{\mathrm{d}s} = -A(x_a - x)\sin\theta. \tag{4}$$

Dividing both members of eqn (4) by $\sin\theta \neq 0$ and differentiating again the new resulting expression once with respect to s, we derive the following strongly non-linear differential equation of the third order (with constant coefficients):

$$\theta'''\sin\theta - \theta'\theta''\cos\theta = A\sin^2\theta\cos\theta. \tag{5}$$

This equation was first presented by Heinzerling (1938) and, since then, was solved in a closed form only for special cases when the bar is submitted to terminal concentrated forces and couples (Timoshenko and Gere, 1961; Love, 1944; Frisch-Fray, 1962; Griner, 1984). It is obvious that, if we consider the same problem, but based on the linear buckling theory (second-order theory), we have

$$\mathrm{d}x = \mathrm{d}s; \quad \mathrm{d}y = \theta\,\mathrm{d}s; \quad \mathrm{d}y/\mathrm{d}x = \theta$$

and consequently eqn (4) becomes

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -A(l-x)\frac{\mathrm{d}y}{\mathrm{d}x}$$

which is in coincidence with formula (b), p. 101, of Timoshenko and Gere (1961).

We shall prove now that eqn (5), by applying a suitable functional transformation, can be reduced to a new equivalent non-linear differential equation of second order. This transformation is given below.

2.1. *Transformation* (i)

We put

$$\theta' = \mathrm{d}\theta/\mathrm{d}s = [\eta(\xi)]^{1/2}; \quad \xi = \cos\theta; \quad \theta = \theta(s) \tag{6}$$

which yields

$$\theta'' = \mathrm{d}(\eta^{1/2})/\mathrm{d}s = [\mathrm{d}(\eta^{1/2})/\mathrm{d}\xi][\mathrm{d}\xi/\mathrm{d}\theta][\mathrm{d}\theta/\mathrm{d}s] = -\tfrac{1}{2}\overset{*}{\eta}\sin\theta \tag{7}$$

$$\theta''' = \tfrac{1}{2}(\overset{**}{\eta}\sin^2\theta - \overset{*}{\eta}\cos\theta)\eta^{1/2} \tag{8}$$

where asterisks indicate differentiation with respect to ξ. In this case eqn (5) is transformed to its equivalent

$$\eta^{1/2}\ddot{\eta} = A\frac{2\xi}{(1-\xi^2)^{1/2}} \quad (\xi/(1-\xi^2)^{1/2} = \cot\theta) \tag{9}$$

which, further, in dimensionless form can be written as

$$(\eta l^2)^{**} - Al^3\frac{2\xi}{(1-\xi^2)^{1/2}}\frac{1}{(\eta l^2)^{1/2}} = Q, \quad \eta l^2 \neq 0; \quad \xi\in[0,1]. \tag{10}$$

The aim of this paper is, first, to present a closed-form solution of the differential equation, eqn (9) (or eqn (10)), for the values of the variable θ lying inside the interval [0°, 20°], which is of practical interest in engineering structures. This solution approximately satisfies the aforementioned buckling problem with a maximum error of less than 2.5%.

In order to transform eqn (9) (or eqn (10)) into an integrable form we shall use successive functional transformations based on a simplified expression for the quantity $\xi/(1-\xi^2)^{1/2}$ or $\cot\theta$ inside the interval [0°, 20°]. We would also underline here that, as will be seen in Section 5, similar approximations of either function $\xi/(1-\xi^2)^{1/2}$ or function $1/(\eta l^2)^{1/2}$ will take place in the other two subintervals for values of θ between (20°, 45°) and [45°, 90°], so that the general solution of the buckling problem is not unique for the whole interval [0°, 90°], but it is divided into three subintervals.

3. ANALYSIS

We shall try to find solutions of eqn (9) (or eqn (10)) inside the interval [0°, 20°] of the slope $\theta(s)$. For this case one may use the approximate relationship

$$\xi = \cos\theta \cong 1-\theta^2/2 \Rightarrow \theta = \sqrt{2}(1-\xi)^{1/2} \tag{11}$$

which is sufficiently accurate for the interval $\theta\in[0°, 20°]$.

Then

$$\frac{\xi}{(1-\xi^2)^{1/2}} = \frac{1-\theta^2/2}{\theta(1-\theta^2/6)} \cong \frac{1}{\theta} = \frac{1}{\sqrt{2}(1-\xi)^{1/2}}. \tag{12}$$

Introducing relation (12) into eqn (10) we obtain the differential equation

$$(\eta l^2)^{1/2}(\eta l^2)^{**} = \sqrt{2}Al^3(1-\xi)^{-1/2}. \tag{13}$$

Then we have already succeeded in transforming the differential equation, eqn (10), to an equivalent form the right-hand member of which includes the monomial $(1-\xi)^{-1/2}$. As we shall see, the equation of this type can be reduced to an equivalent first-order non-linear differential equation. For this purpose the following functional transformations are required.

3.1. *Transformation* (i)

We introduce, instead of $\eta(\xi)$, another function $p(\omega)$ of a new independent variable ω by taking

$$\eta(\xi)l^2 = tp(\omega); \quad \omega = \ln t; \quad t = 1-\xi. \tag{14}$$

In this case one may get

$$[\eta(\xi)l^2]^* = \mathrm{d}(\eta l^2)/\mathrm{d}\xi = \{[tp(\omega)]/\mathrm{d}\omega\}(\mathrm{d}\omega/\mathrm{d}t)(\mathrm{d}t/\mathrm{d}\xi) = -(tp+t\dot{p})/t = -(p+\dot{p}) \quad (15)$$

$$[\eta(\xi)l^2]^{**} = (\dot{p}+\ddot{p})/t \quad (16)$$

where dots denote differentiation with respect to ω.

Inserting eqns (14)–(16) into eqn (13) we find

$$\ddot{p}+\dot{p} = \sqrt{2Al^3}p^{-1/2}. \quad (17)$$

The following transformation.

3.2. *Transformation* (ii)

$$\dot{p}(\omega) = h(r)-p; \quad \mathrm{d}r = \sqrt{2Al^3}p^{-1/2}\,\mathrm{d}p \quad (p = r^2/(2\sqrt{2Al^3})^2) \quad (18)$$

and the expression

$$\ddot{p} = \frac{\mathrm{d}}{\mathrm{d}\omega}(h-p) = \frac{\mathrm{d}}{\mathrm{d}p}(h-p)\frac{\mathrm{d}p}{\mathrm{d}\omega} = [(\mathrm{d}h/\mathrm{d}p)-1](h-p)$$
$$= [(\mathrm{d}h/\mathrm{d}r)(\mathrm{d}r/\mathrm{d}p)-1](h-p) = (h'\sqrt{2Al^3}p^{-1/2}-1)(h-p)$$

introduced into eqn (17) yields the following first-order Abel equation:

$$(h-p)h' = 1. \quad (19)$$

Here primes represent differentiation with respect to the variable r. By inverting the differentials in the last equation one may obtain

$$\mathrm{d}r/\mathrm{d}h = h-k^2r^2; \quad k^2 = 1/(2\sqrt{2Al^3})^2 \quad (20)$$

which is a Riccati differential equation with respect to r.

From now on, using the well-known transformation given on p. 22 of Kamke (1971)

$$r(h) = \frac{\overset{*}{u}}{k^2u}; \quad \overset{*}{r} = \frac{u\overset{**}{u}-\overset{*}{u}{}^2}{k^2u^2} \quad (21)$$

where stars indicate differentiations with respect to h, the Riccati equation (20) leads to the following second-order linear differential equation of the Bessel type (case 10 on p. 440 of Kamke (1971)):

$$\overset{**}{u}-k^2hu = 0 \quad (22)$$

the general integral of which is given by

$$u(h) = \sqrt{h}[c_1J_{1/3}(2ikh^{3/2}/3)+c_2N_{1/3}(2ikh^{3/2}/3)]. \quad (23)$$

In this relation $i = \sqrt{(-1)}$; $J_{1/3}$, $N_{1/3}$ are Bessel functions of the first and second kind, respectively, of order 1/3; and, c_1; c_2 represent random constants of integration.

We notice here that the reduction of the non-linear differential equation, eqn (10), to a Bessel equation, eqn (22), verifies the contentions of Greenhill (1884), Jasinsky (1902), Dondorff (1907) and Kàrmàn and Biot (1940), according to which a variety of buckling problems can be solved by using Bessel functions.

4. SOLUTION TECHNIQUE

It is obvious from the above-mentioned procedure of the solution of eqn (19) (or eqn (20)), that the inverse course, i.e. the determination of the function $\theta(s)$ through closed-form solution (23) is a complicated problem, because multiple inversions of the Bessel functions are required. In spite of these, as we shall prove, general solution (23) can be conveniently approached in the considered interval [0°, 20°] so as to obtain a closed-form expression of the $\theta(s)$-function.

In fact, the general solution (23), after a little algebra, can be written in the form

$$u(h) = ch\left(1+\frac{k^2h^3}{12}+\frac{k^4h^6}{504}+\cdots\right)+\mathring{c}\left(1+\frac{k^2h^3}{6}+\frac{k^4h^6}{180}+\cdots\right) \tag{24}$$

where c; $\mathring{c}$ are new constants of integration.

We introduce now the boundary conditions

$$\text{for}\quad x = x_\alpha \Rightarrow \theta_\alpha = \alpha;\quad \theta'_\alpha = 0.$$

Observing from relation (4) that $\theta''_\alpha = 0$ when $x = x_\alpha$, and using functional transformations (1)–(12) we may derive that

$$\theta'^2_\alpha = \eta_\alpha = p_\alpha = r_\alpha = \mathring{u}_\alpha = 0.$$

On the other hand, because of the validity of equations

$$\dot{p}(\omega) = h(r)-p;\quad \dot{p} = -(\eta l^2)^*-p = [(2\theta'' l^2)/\sin\theta]-p$$

we obtain

$$\dot{p}_\alpha = 0;\quad h_\alpha = 0.$$

Combining the previous results together with solution (24) we obtain $\mathring{c} = 0$, and therefore, the general solution of the problem takes the form

$$u(h) = ch\left(1+\frac{k^2h^3}{12}+\frac{k^4h^6}{504}+\cdots\right). \tag{25}$$

In the sequel we shall prove that for every θ lying inside the interval [0°, 20°], the terms $k^2h^3/12$; $k^4h^6/504$; ... are very small in comparison with unity, so that they can be neglected. In fact, supposing that the general solution $u(h)$ is expressed by

$$u(h) = ch \tag{26}$$

then, using functional transformation (21), we have

$$r(h) = \frac{1}{k^2h} \Leftrightarrow h = \frac{1}{k^2r}. \tag{27}$$

From now on we can readily prove the inequality

$$k^2h^3/12 \ll 1$$

or, based on the last equation, the inequality

$$1/12k^4r^3 \ll 1. \tag{28}$$

From relations (18); (14); (11) and (6) one may obtain

$$r^2 = p\sqrt{p/k^3}; \quad p = \eta l^2/(1-\xi) = 2\eta l^2/\theta^2 = 2(\theta' l)^2/\theta^2 = 2(\kappa l)^2/\theta^2$$

in which $\kappa = \kappa(s)$ denotes the curvature function of the bar. Consequently, inequality (28), after some algebra, leads to

$$(\theta/\kappa l)^3(Al^3)/12 = (\theta/\kappa l)^3(ql^3/12EJ) \ll 1.$$

For example, for a prismatic bar of length $l = 300$ cm made of steel, with a quadrangular cross-section $F = 5 \times 5\ \mathrm{cm}^2$, its own weight is equal to 0.196 kp cm^{-1} and therefore the last inequality becomes

$$(\theta/\kappa l)^3 \times 4.08 \times 10^{-3} \ll 1.$$

But, in the classical problem of elastica in cantilevers, the quantity κl can be replaced by the equivalent expression (p. 78 of Timoshenko and Gere (1961))

$$\kappa l = \sqrt{2}K[\sin(\alpha/2)](\cos\theta - \cos\alpha)^{1/2}$$

in which $K[\sin(\alpha/2)]$ is the complete elliptic integral of the first kind of modulus $\sin(\alpha/2)$. Considering the maximum angle α of the free end of the cantilever (in our case $\alpha = 20°$) and an arbitrary value of the angle θ inside the interval [0°, 20°], the validity of the last inequality is obvious. Using now eqns (21) and (18) we lead to the integral

$$k\sqrt{p}/(1 - kp\sqrt{p}) = \mathrm{d}\omega \tag{29}$$

which gives

$$1 - kp\sqrt{p} = c\, \mathrm{e}^{-3\omega/2} \tag{30}$$

where c is a new constant of integration.

Combining relations (30) and (14) together with eqn (11) we obtain

$$(1-\cos\theta)^{3/2} - k(\eta l^2)(\eta l^2)^{1/2} = c.$$

Using the half angle identity $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$ and rearranging, we obtain

$$8\sin^3(\theta/2) - \theta'^3/A = c. \tag{31}$$

Observing that $\theta'_\alpha = 0$ when $\theta_\alpha = \alpha$, we conclude from the last equation that

$$c = \sin^3(\alpha/2)$$

and hence

$$\frac{\mathrm{d}\theta}{[\sin^3(\theta/2) - \sin^3(\alpha/2)]^{1/3}} = 2A^{1/3}\,\mathrm{d}s. \tag{32}$$

Taking the integrals in both members of eqn (32) we have

$$-2A^{1/3}\int_0^l ds = -A^{1/3}\int_{-l}^{l} ds = \int_{-\alpha}^{\alpha} d\theta/[\sin^3(\theta/2)-\sin^3(\alpha/2)]^{1/3}. \tag{33}$$

This elliptic integral can be simplified by using the notation $p = \sin(\alpha/2)$ and by introducing a new variable φ in such a manner that

$$\sin(\theta/2)/\sin(\alpha/2) = \sin(\theta/2)/p = \sin\varphi. \tag{34}$$

It is seen from this relation that when θ varies between $-\alpha$ and α, $\sin\varphi$ varies between -1 and 1; hence φ varies from $-\pi/2$ to $\pi/2$. We also find from eqn (34), by differentiation, that

$$d\theta = 2p\cos\varphi\, d\varphi/\cos\theta = 2p\cos\varphi\, d\varphi/(1-p^2\sin^2\varphi)^{1/2}.$$

Substituting in eqn (33) and approximating the square root we obtain

$$-A^{1/3}l = \int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi/(\sin^3\varphi-1)^{1/3}+(p^2/2)\int_{-\pi/2}^{\pi/2}\cos\varphi\sin^2\varphi\, d\varphi/(\sin^3\varphi-1)^{1/3}. \tag{35}$$

The second integral of this relation gives

$$(p^2/6)\int_{-\pi/2}^{\pi/2} d(\sin^3\varphi)/(\sin^3\varphi-1)^{1/3} = -p^2 4^{-1/3}/4 = -0.397p^2 \tag{36}$$

while the first integral takes the form

$$\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi/(\sin^3\varphi-1)^{1/3}$$
$$= \int_{-\pi/2}^{\pi/2}\sin\varphi\, d\varphi/(\sin\varphi-1)\{[\sin\varphi-(-1/2)]^2+(\sqrt{3}/2)^2\}^{1/3}. \tag{37}$$

We now make use of the substitution (p. 160 of Byrd and Friedman (1971))

$$\sin\varphi = (\sqrt{3}\tau-3)/(\sqrt{3}\tau+3);\quad \cos\varphi\, d\varphi = 6\sqrt{3}\, d\tau/(\sqrt{3}\tau+3)^2;\quad \tau\in[0,\infty]$$

and integral (37) is transformed to the equivalent

$$J = -\frac{2^{2/3}}{\sqrt{3}}\int_0^{\infty}\frac{d\tau}{[1+(\tau/\sqrt{3})](\tau^2+1)^{1/3}}. \tag{38}$$

The trapezoidal quadrature rule is ideal for the numerical computation of this class of elliptic integrals. Making these calculations one may conclude that the above integral converges to the value $|J| = 2.000$. Therefore, relation (33) leads to

$$2.000+0.397p^2 = A^{1/3}l \tag{39}$$

from which one may conclude that the value of the uniform load for the bar shown in Fig. 1 (depending on the value of the angle α at the top) is given by

$$ql = (2.000 + 0.397p^2)^3 EJ/l^2. \quad (40)$$

We notice here that, when the deflection of the bar is very small, it is valid $\alpha = 0°$ and from eqn (40) we obtain

$$(ql)_{cr} = 8.000EJ/l^2$$

which is the value of the critical load (with an error of about 2%) according to the linear buckling theory (second-order theory).

As the value of α increases from 0° to 20° the quantity ql also increases. For example, take the case where $\alpha = 20°$ and $p = \sin(\alpha/2) = 0.174$. Using relation (39) we evaluate for this case

$$ql = 8.145EJ/l^2.$$

Taking the ratio of ql to the critical load, we find

$$ql/(ql)_{cr} = 8.145/8.000 = 1.018.$$

Thus, a load which is 1.8% greater than the Euler load, at which buckling first begins, will produce a deflection such that the tangent at the top subtends an angle of 20° with the vertical. This result is analogous to the result given in Table 2-4, p. 79 of Timoshenko and Gere (1961), for the case of the classical problem of elastica in cantilevers, due exclusively to terminal concentrated loading.

5. GENERALIZATIONS AND DISCUSSION

In the previous sections we have succeeded in giving closed-form solutions of the strongly non-linear differential equations describing the buckling problem in cantilevers, due to axial uniformly distributed loads. These solutions were derived by using elliptic integrals and were valid for the values of the slope θ of the deflected elastica lying inside the interval [0°, 20°]. We also proved that the formula giving the critical load (relation (40)) is divided into two terms; the first term is in accordance with the corresponding buckling load for the linear theory (second-order theory), while the second term depends on the free-end-slope of the deflected elastica.

Here, as an extension of the proposed methodology, we shall try to reduce the differential equation, eqn (9) (or eqn (10)), to simple integrable forms in other subintervals inside the remaining interval [20°, 90°]. Since the previous reduction was based on the approximate expression of the function $\xi/(1-\xi^2)^{1/2}$ (or $\cot\theta$) in the subinterval [0°, 20°], we try now to define similar approximations in the subsequent subintervals. The following cases are distinguished.

5.1. *Case* (i) : $\theta \in (20°, 45°)$

Using the graph of $\xi/(1-\xi^2)^{1/2}$; $\xi \in [0, 1]$, one may conclude that this function can be approximately expressed by a polynomial of the second degree of the form

$$\xi/(1-\xi^2)^{1/2} \simeq \alpha\xi^2 + \beta\xi + \gamma \quad (\alpha = 19; \beta = -25; \gamma = 9.17).$$

On the other hand, based on the methodology developed in Section 4, one may prove that the function $1/(\eta l^2)^{1/2} = 1/\kappa l$ (κ is the curvature function) can be written as

$$1/(\eta l^2)^{1/2} = 1/[1+(\eta l^2 - 1)]^{1/2} \simeq 1-(\eta l^2-1)/2 = (3-\eta l^2)/2.$$

Consequently, the differential equation, eqn (9), takes the simpler form

$$(3-\eta l^2)^{**}+A(\alpha\xi^2+\beta\xi+\gamma)(3-\eta l^2) = 0. \tag{41}$$

This form of the differential equation is included on p. 417, 2.55 of Kamke (1971) and the functional transformation for its further reduction is expressed by

$$3-\eta l^2 = u(\xi)\, e^{i(A\alpha)^{1/2}\xi^2/2}. \tag{42}$$

In fact, using relation (42), eqn (41) is transformed into

$$\ddot{u}+\lambda\xi\dot{u}+(\beta\xi+\delta)u = 0; \quad [\lambda = 2i(A\alpha)^{1/2}, \quad \delta = \gamma+i(A\alpha)^{1/2}] \tag{43}$$

which, by the substitution

$$u(\xi) = y(x) = e^{-\beta\xi/\lambda}; \quad x = \sqrt{|\lambda|}(\xi-2\beta/\lambda^2) \tag{44}$$

becomes

$$y''+xy+i(\beta^2-4A\alpha)y/8(A\alpha)^{3/2} = 0 \tag{45}$$

(primes denote differentiation with respect to x).

Equation (45) is a differential equation of the Whittaker type and its closed-form solution is included on p. 473, 2.273, type (10) of Kamke (1971). We notice that in this case the solution of eqn (9) is achieved by the Whittaker functions, namely according to a procedure analogous to the methodology followed for the values of the slope θ lying inside the interval $[0^\circ, 20^\circ]$.

5.2. *Case* (ii) : $\theta \in [45^\circ, 70^\circ)$ *and Case* (iii) : $\theta \in [70^\circ, 90^\circ]$

In these two cases the differential equation, eqn (9), can be simplified as

$$(\eta)^{1/2}\ddot{\eta} = 2A(\alpha\xi+\beta) \tag{46}$$

in which

$$\alpha = 1.55; \quad \beta = -0.18 \quad \text{for} \quad \theta \in [45^\circ, 70^\circ)$$
$$\alpha = 1.00; \quad \beta = 0.00 \quad \text{for} \quad \theta \in [70^\circ, 90^\circ].$$

Making use of the substitution

$$\eta(\xi) = p(\omega); \quad \alpha\xi+\beta = \omega \tag{47}$$

the last differential equation is transformed to

$$(p)^{1/2}\ddot{p} = 2A\omega/\alpha^2 \tag{48}$$

where dots represent differentiation with respect to ω.

This differential equation is of the same type as eqn (13). Then, if we apply transformations 3.1 and 3.2, the resulting Abel equation cannot be integrated, as in the case of Section 3. This is the reason why we have introduced the following new analytical treatment.

Multiplying both members of eqn (48) by $[\dot{p}/(\eta)^{1/2}-4A\omega/\alpha^2]$ (see p. 586, type 6.190—Euler's method of Kamke (1971)), we have

$$(\dot{p}^3)^{\cdot}-12A\omega(p)^{1/2}\ddot{p}/\alpha^2-6A\omega\dot{p}^2/\alpha^2(p)^{1/2}+8\alpha^2[(A\omega/\alpha^2)^3]^{\cdot}/A = 0$$

which can be further written as

$$(\dot{p}^3)^{\cdot} - 12A[\omega(p)^{1/2}\dot{p}]^{\cdot}/\alpha^2 + 8A[(p^3)^{1/2}]^{\cdot}/\alpha^2 + 8\alpha^2[(A\omega/\alpha^2)^3]^{\cdot}/A = 0. \tag{49}$$

Integrating we obtain

$$\dot{p}^3 - 12A\omega(p)^{1/2}\dot{p}/\alpha^2 + 8A(p^3)^{1/2}/\alpha^2 + 8A^2\omega^3/\alpha^4 = c_1 \tag{50}$$

where c_1 is an integration constant.

By now, using functional transformations (47) and (6), eqn (50) leads to the following first-order strongly non-linear differential equation with respect to η:

$$-(\dot{\eta}/\sin\theta)^3 + 12A(\alpha\cos\theta+\beta)(\eta)^{1/2}(\dot{\eta}/\sin\theta) + 8A\alpha(\eta^3)^{1/2} + 8A^2(\alpha\cos\theta+\beta)^3/\alpha - c_1 = 0. \tag{51}$$

Here dots denote differentiations with respect to θ.

We introduce the boundary conditions

$$\text{for}\quad x = x_\alpha \Rightarrow \theta_\alpha = \alpha;\quad \theta'_\alpha = (\eta)_\alpha^{1/2} = 0;\quad \theta''_\alpha = (\dot{\eta})_\alpha/2 = 0.$$

If θ lies inside the interval [45°, 70°), then max $\alpha = 70°$, while if θ lies inside the interval [70°, 90°], then max $\alpha = 90°$. Combining these boundary conditions with eqn (51) we conclude that, for the two values of θ, the constant of integration c_1 becomes respectively

$$c_1 = 8A^2(\alpha\cos\alpha+\beta)^3/\alpha;\quad c_1 = 8A^2\cos^3\alpha.$$

Consequently, for the two cases, the last three terms of eqn (51) take the forms

$$8A\alpha\{(\eta^3)^{1/2} + A[(\alpha\cos\theta+\beta)^3 - (\alpha\cos\alpha+\beta)^3]/\alpha^2\};\quad 8A[(\eta^3)^{1/2} + A(\cos^3\theta - \cos^3\alpha)].$$

By using the results of the classical problem of elastica in cantilevers (pp. 77–78 of Timoshenko and Gere (1961)), it is easy to prove that the terms $A[(\alpha\cos\theta+\beta)^3 - (\alpha\cos\alpha+\beta)^3]/\alpha^2$ and $A(\cos^3\theta - \cos^3\alpha)$ are negligible in comparison with $(\eta^3)^{1/2} = \theta'^3$, so that they can be rejected. So, eqn (51) is written as

$$-\dot{\eta}^3 + 12A(\alpha\cos\theta+\beta)\sin^2\theta\dot{\eta} + 8A\alpha(\eta^3)^{1/2}\sin^3\theta = 0. \tag{52}$$

Relation (52) is an algebraic equation of Cardan's form with respect to $[\dot{\eta}/(\eta)^{1/2}]$. Putting

$$P = -\frac{12A\sin^2\theta(\alpha\cos\theta+\beta)}{(\eta)^{1/2}};\quad Q = -8A\alpha\sin^3\theta \tag{53}$$

one may observe that the discriminant

$$D = (P/3)^2 + (Q/2)^3$$

is positive. Therefore, the first positive root of eqn (52) becomes

$$\dot{\eta}/(\eta)^{1/2} = -2(-P/3)^{1/2}\operatorname{cosec}2\varphi;\quad \tan\varphi = [\tan(\beta/2)]^{1/3};\quad \sin\beta = 2(-P/3)^{3/2}/Q. \tag{54}$$

Introducing the quantity P from the first of eqns (54) we conclude that the angles β and φ are very small so that one may write

$$\sin\beta \simeq \beta;\quad \tan\varphi \simeq \varphi;\quad \operatorname{cosec}\varphi \simeq (1/2\varphi) + (\varphi/3).$$

Therefore, eqn (52) is reduced to

$$\dot{\eta} = B \sin \theta(\eta)^{1/2} + \Gamma \sin \theta(\alpha \cos \theta + \beta); \quad B = 2A^{1/3}; \quad \Gamma = 4A^{2/3}/3. \tag{55}$$

Then it is valid

$$\dot{\eta} = 2(\eta)^{1/2}[(\eta)^{1/2}]^{\cdot}$$

and the last equation takes the form

$$(\eta)^{1/2}[(\eta)^{1/2}]^{\cdot} = \overset{\bullet}{B} \sin \theta(\eta)^{1/2} + \overset{\bullet}{\Gamma} \sin \theta(\alpha \cos \theta + \beta); \quad \overset{\bullet}{B} = B/2; \quad \overset{\bullet}{\Gamma} = \Gamma/2 \tag{56}$$

which is an Abel equation with respect to $(\eta)^{1/2}$. If θ lies inside the interval [70°, 90°], then $\alpha = 1$; $\beta = 0$ and eqn (56) is transformed into

$$(\eta)^{1/2}[(\eta)^{1/2}]^{\cdot} = \overset{\bullet}{B} \sin \theta(\eta)^{1/2} + \overset{\bullet}{\Gamma} \sin \theta \cos \theta. \tag{57}$$

We shall prove now that eqn (57) can be reduced to a linear second-order ordinary differential equation with constant coefficients. In fact, putting

$$(\eta)^{1/2} = u(\theta) - \overset{\bullet}{B} \cos \theta \tag{58}$$

we have

$$[(\eta)^{1/2}]^{\cdot} = \mathrm{d}(u - \overset{\bullet}{B} \cos \theta)/\mathrm{d}\theta = \dot{u} + \overset{\bullet}{B} \sin \theta$$

and hence

$$u - \overset{\bullet}{B} \cos \theta = \overset{\bullet}{\Gamma} \sin \theta \cos \theta \, \theta' \tag{59}$$

where primes denote differentiation with respect to u. Using the new transformation

$$1 - \sin^2 \theta = y^2(u) \tag{60}$$

leads to the equation

$$-\overset{\bullet}{\Gamma} y y' + \overset{\bullet}{B} y = u \tag{61}$$

which, further, by putting

$$y(u) = 1/g' = 1/(\mathrm{d}g/\mathrm{d}u) = \mathrm{d}u/\mathrm{d}g = \dot{u}, \quad (y' = \ddot{u}/\dot{u})$$

gives

$$\overset{\bullet}{B}\ddot{u} - \overset{\bullet}{\Gamma}\dot{u} + u = 0. \tag{62}$$

Here dots represent differentiation with respect to g.

For the second case, i.e. if θ lies inside the interval (45°, 70°), the corresponding equation to eqn (59) becomes

$$u - \overset{\bullet}{B} \cos \theta = \alpha \overset{\bullet}{\Gamma} \sin \theta \cos \theta \, \theta' + \beta \overset{\bullet}{\Gamma} \sin \theta \, \theta'. \tag{63}$$

Using the appropriate transformation (60) we lead to the differential equation

$$y y' + \beta y'/\alpha = \overset{\bullet}{B} y/\overset{\bullet}{\Gamma}\alpha - u/\overset{\bullet}{\Gamma}\alpha. \tag{64}$$

Putting

$$z(u) = y(u) + \beta/\alpha$$

we obtain

$$zz' - \dot{B}z/\dot{\Gamma}\alpha + u/\dot{\Gamma}\alpha + \beta\dot{B}/\alpha^2\dot{\Gamma} = 0 \tag{65}$$

where primes denote differentiation with respect to u.

Using the final transformation

$$z = 1/g' = 1/(\mathrm{d}g/\mathrm{d}u) = \mathrm{d}u/\mathrm{d}g = \dot{u}, \quad (z' = \ddot{u}/\dot{u})$$

eqn (65) is reduced to the following complete second-order ordinary differential equation with respect to $u(g)$

$$\ddot{u} - \dot{B}\dot{u}/\dot{\Gamma}\alpha + u/\dot{\Gamma}\alpha + \beta\dot{B}/\alpha^2\dot{\Gamma} = 0 \tag{66}$$

(dots represent differentiation with respect to g). We notice here that both eqns (62) and (66) can be integrated by closed-form relations.

From the above analysis it becomes clear that the elastica configuration of the bar in the subintervals of θ considered cannot be described by a unique solution. If more than one of the previous solutions must be used, then, a number of compatibility conditions must be also introduced. These conditions result from the equality of the deflections and slopes at the common points of the neighbouring subintervals, and are sufficient for the evaluation of the integration constants.

6. CONCLUSIONS

This paper establishes a closed-form solution for the problem of the elastica analysis for straight and prismatic bars due to an axial distributed load along its length.

The theory developed was based:

(i) on the introduction of convenient functional transformations, which succeed in reducing the arising complicated strongly non-linear differential equation of the slope to simple integrable forms;

(ii) on the development of a quantitative analysis in accordance with the physical problem necessary for a more accurate solution of the aforementioned governing differential equation.

Also, in the analysis, it was proved that:

(i) for slopes lying inside the interval [0°, 20°], the formula giving the critical load is divided into two terms; the first is in accordance with the corresponding buckling load in linear theory, while the second term depends on the free-end slope of the deflected elastica;

(ii) for slopes lying inside the subintervals included in the interval (20°, 90°] the non-linear equilibrium differential equation can be reduced to other integrable forms.

Consequently, it is derived from the above analysis that the elastica configuration into the interval [0°, 90°] of the slope cannot be described by a unique solution.

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